

Relativistic quantum dynamics of vector bosons in an Aharonov–Bohm potential

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Abstract The Aharonov–Bohm (AB) problem for vector bosons by the Duffin–Kemmer–Petiau (DKP) formalism is analyzed. Depending on the values of the spin projection, the relevant eigenvalue equation coming from the DKP formalism reveals an equivalence to the spin-1/2 AB problem. By using the self-adjoint extension approach, we examine the bound state scenario. The energy spectra are explicitly computed as well as their dependencies on the magnetic flux parameter and also the conditions for the occurrence of bound states.

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1 Introduction

The Aharonov–Bohm (AB) effect [1] has been an usual framework for investigating the arising of phases in the wave function of quantum particles in various physical models and has inspired a great deal of investigations in recent years. In the AB effect, the vector potential due to a solenoid gains an extraordinary physical meaning. It can affect the quantum behavior of a charged particle that never encounters an electromagnetic field. This phenomenon is intimately related to a non-local boundary condition which relates the change in the phase of an electron wave function to the amount of flux in the solenoid. The interest in this issue appears in the different contexts, such as solid state physics [2], cosmic strings [3–11] κ -Poincaré–Hopf algebra [12, 13], δ -like singularities [14], supersymmetry [15], condensed matter [16], Lorentz symmetry violation [17–19], quantum chromodynamics [20], general relativity [21], nanophysics [22], quantum ring [23–26], black hole [27] and noncommutative theories [28–30].

In the AB problem of spin-1/2 particles a two-dimensional δ -function appears as the mathematical description of the Zeeman interaction between the spin and the magnetic flux tube [14]. This interaction term is known to cause a splitting on the energy spectrum of atoms depending on the spin state. In AB problem of spin-1 particles [31, 32], however, this characteristic is also present. In Ref. [31], where the authors address the AB problem for spin-1 Yang–Mills particles, it was established that, for the case of spin-1/2, quasibound states exist for all noninteger flux parameter. The existence of these states is related to the penetration of the magnetic flux tube by the particle, which is sufficient to produce sensitivity to the sign of the flux. The difference for spin-1 Yang–Mills particles is that the quasibound states exist only for discrete values of the magnetic flux tube, so that penetration occurs only for flux values in a set of measure zero.

In this work, we solve the spin-1 AB problem for bound states in the context of the DKP formalism. In our approach, we consider the idealized picture of a magnetic flux tube of null radius which allows the particles to access the $r = 0$ region in a controlled way. Unlike the approach taken in Ref. [31], here, we modulate the problem with general boundary conditions. When the spin projection $s^3 = 0$, the radial operator can be expressed as a modified Bessel differential equation. In this case, the system does not admit bound-state solutions. On the other hand, when the spin projection $s^{1,2} = 1, -1$, as we mentioned above, we have the presence of a δ -function potential in the equation of motion. As is well-known in quantum mechanics, the δ -function potential guarantees at least one bound state for the particle and this property is independent of its spin. For the system considered here, in first sight, the inclusion of the spin projection element $s^{1,2}$ leads to an equation of motion equivalent to the equation for the spin-1/2 AB problem. Because of this, the problem can be addressed by the self-adjoint extension

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method [33, 34] with the application of appropriate boundary conditions. After imposing the boundary conditions, we finally determine the bound states of the vector bosons in terms of the physics of the problem, in a very consistent way and without any arbitrary parameter.

2 A short review on Duffin–Kemmer–Petiau equation

The first-order DKP formalism [35–38] describes spin-0 and spin-1 particles and has been used to analyse relativistic interactions of spin-0 and spin-1 hadrons with nuclei as an alternative to their conventional second-order Klein–Gordon (KG) and Proca counterparts. Although the formalisms are equivalent in the case of minimally coupled vector interactions [39–41], the DKP formalism enjoys a richness of couplings not capable of being expressed in the KG and Proca theories [42, 43]. Indeed, the DKP formalism has been widely used in the description of many processes in elementary particle and nuclear physics and it proved to be better than the KG formalism in the analysis of K_{l3} decays, the decay-rate ratio $\Gamma(\eta \rightarrow \gamma\gamma)/\Gamma(\pi^0 \rightarrow \gamma\gamma)$, and level shifts and widths in pionic atoms [44–46]. The DKP formalism has also applications in other contexts, as such, in noncommutative phase space [47], in Very Special Relativity (VSR) symmetries [48], in Bose–Einstein condensates [49, 50], in topological defects [51], in thermodynamics properties [52], in topological semimetals [53], in noninertial effect of rotating frames [54], among others.

The DKP equation for a free charged boson is given by [38] (with units in which $\hbar = c = 1$)

$$(i\beta^\mu \partial_\mu - M) \Psi = 0, \quad (1)$$

where the matrices β^μ satisfy the DKP algebra

$$\beta^\mu \beta^\nu \beta^\lambda + \beta^\lambda \beta^\nu \beta^\mu = g^{\mu\nu} \beta^\lambda + g^{\lambda\nu} \beta^\mu, \quad (2)$$

and the metric tensor is $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$. That algebra generates a set of 126 independent matrices whose irreducible representations are a trivial representation, a five-dimensional representation describing the spin-0 particles and a ten-dimensional representation associated to spin-1 particles. The DKP spinor has an excess of components and the theory has to be supplemented by an equation which allows to eliminate the redundant components. That constraint equation is obtained by multiplying the DKP equation by $1 - \beta^0 \beta^0$, namely

$$i\beta^j \beta^0 \partial_j \Psi = M(1 - \beta^0 \beta^0) \Psi, \quad (3)$$

where j runs from 1 to 3. This constraint equation expresses three (four) components of the spinor by the other two (six) components and their space derivatives in the scalar (vector) sector so that the superfluous components disappear and

there only remain the physical components of the DKP theory. The second-order KG and Proca equations are obtained when one selects the spin-0 and spin-1 sectors of the DKP theory. A well-known conserved four-current is given by

$$J^\mu = \frac{1}{2} \bar{\Psi} \beta^\mu \Psi, \quad (4)$$

where the adjoint spinor $\bar{\Psi}$ is given by $\bar{\Psi} = \Psi^\dagger \eta^0$ with $\eta^0 = 2\beta^0 \beta^0 - 1$ in such a way that $(\eta^0 \beta^\mu)^\dagger = \eta^0 \beta^\mu$ (the matrices β^μ are Hermitian with respect to η^0). Despite the similarity to the Dirac equation, the DKP equation involves singular matrices, the time component of J^μ given by (4) is not positive definite and the case of massless bosons can not be obtained by a limiting process [55]. Nevertheless, the matrices β^μ plus the unit operator generate a ring consistent with integer-spin algebra and J^0 may be interpreted as a charge density. The factor $1/2$ multiplying $\bar{\Psi} \beta^\mu \Psi$, of no importance regarding the conservation law, is in order to hand over a charge density conformable to that one used in the KG theory and its nonrelativistic limit [56].

3 Interactions in the Duffin–Kemmer–Petiau equation

With the introduction of interactions, the DKP equation can be written as

$$(i\beta^\mu \partial_\mu - M - U) \Psi = 0, \quad (5)$$

where the more general potential matrix U is written in terms of 25 (100) linearly independent matrices pertinent to five (ten)-dimensional irreducible representation associated to the scalar (vector) sector. In the presence of interaction, J^μ satisfies the equation

$$\partial_\mu J^\mu + \frac{i}{2} \bar{\Psi} (U - \eta^0 U^\dagger \eta^0) \Psi = 0. \quad (6)$$

Thus, if U is Hermitian with respect to η^0 , then four-current will be conserved. The potential matrix U can be written in terms of well-defined Lorentz structures. For the spin-0 (scalar sector) there are two scalar, being two vector and two tensor terms [42], whereas for the spin-1 (vector sector) there are two scalar, two vector, a pseudoscalar, two pseudovector and eight tensor terms [43]. The condition (6) has been used to point out a misleading treatment in the recent literature regarding analytical solutions for nonminimal vector interactions [57–59].

3.1 Duffin–Kemmer–Petiau equation with minimal electromagnetic coupling

Considering only the minimal vector interaction, the DKP equation for a charged boson with minimal electromagnetic coupling is given by

$$(i\beta^\mu D_\mu - M) \Psi = 0, \quad (7)$$

where the covariant derivative is given by $D_\mu = \partial_\mu + ieA_\mu$. In this case, the constraint equation (3) becomes

$$i\beta^k\beta^0\beta^0\partial_k\Psi - e\beta^k\beta^0\beta^0A_k\Psi = M(1 - \beta^0\beta^0)\Psi, \quad (8)$$

and the four-current J^μ retains its form as (4).

3.2 Vector sector

Now, we discuss the vector sector (spin-1 sector) of the DKP theory. To select the physical component of the DKP field for the vector sector (spin-1 sector), we define the operator [60]

$$R^\mu = (\beta^1)^2(\beta^2)^2(\beta^3)^2[\beta^\mu\beta^0 - g^{\mu 0}], \quad (9)$$

which satisfies $R^{\mu\nu} = R^\mu\beta^\nu$ and $R^{\mu\nu} = -R^{\nu\mu}$. Moreover, as it is shown in Ref. [60], $R^\mu\Psi$ and $R^{\mu\nu}\Psi$ transform as a (pseudo)vector and (pseudo)tensor quantities under an infinitesimal Lorentz transformation, respectively. From the above definitions, the following property is obtained:

$$R^{\mu\nu}\beta^\alpha = R^\mu g^{\nu\alpha} - R^\nu g^{\mu\alpha}. \quad (10)$$

In this way, by applying the R^μ and $R^{\mu\nu}$ operators to the DKP equation (7), we obtain

$$D_\mu(R^{\nu\mu}\Psi) = -iM(R^\nu\Psi), \quad (11a)$$

$$(R^{\mu\nu}\Psi) = -\frac{i}{M}U^{\mu\nu}, \quad (11b)$$

$$U^{\mu\nu} = D^\mu(R^\nu\Psi) - D^\nu(R^\mu\Psi), \quad (11c)$$

which leads to

$$D_\mu U^{\mu\nu} + M^2(R^\nu\Psi) = 0, \quad (12a)$$

$$D_\mu(R^\mu\Psi) = \frac{ie}{2M^2}F_{\mu\nu}U^{\mu\nu}, \quad (12b)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. These results tell us that all elements of the column matrix $R^\mu\Psi$ obey the Proca equation interacting minimally with an electromagnetic field. So, this procedure selects the vector sector of DKP theory, making explicitly clear that it describes a spin-1 particle embedded in a electromagnetic field.

According to Ref. [61], we can rewrite Eq. (12) in the form

$$[D_\mu D^\mu + M^2]R^\nu\Psi - D^\nu D_\mu R^\mu\Psi - \frac{ie}{2}R^\nu S^{\alpha\mu}F_{\mu\alpha}\Psi = 0, \quad (13)$$

where $S^{\mu\nu} = [\beta^\mu, \beta^\nu]$. The term $D^\nu D_\mu R^\mu\Psi$ is called the anomalous term because it has no equivalent in the spin-1/2 Dirac theory [38]. However, it has been shown in Refs. [39, 40] that such an anomalous term disappears when the physical components of the DKP field are selected.

Following the same procedure of Ref. [61], Eq. (13) becomes

$$[D_\mu D^\mu + M^2 - e(\mathbf{S} \cdot \mathbf{B})]\mathbf{R}\Psi = 0. \quad (14)$$

with $\mathbf{B} = \nabla \times \mathbf{A}$ and the spin operator $\mathbf{S} = (S^1, S^2, S^3)$ is expressed by

$$S^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, S^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, S^3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (15)$$

In this stage, it is worthwhile to mention that the last term in equation (14) is called Pauli term, which is crucial to give meaning to the term that explicitly depends of the spin. Also, we can mention that the Pauli term is only important for the vector sector (spin-1 sector) of the DKP theory, because that term is absent for the scalar sector (spin-0 sector). For this reason, we only focus the vector sector of the DKP theory.

If the terms in the potential $A^\mu = (A_0, \mathbf{A})$ are time-independent one can write

$$\Psi(\mathbf{r}, t) = \psi(\mathbf{r})e^{-iEt}, \quad (16)$$

where E is the energy of the vector boson, in such a way that the time-independent DKP equation for the vector sector becomes

$$[(\mathbf{p} - e\mathbf{A})^2 + M^2 - (E - eA_0)^2 - e(\mathbf{S} \cdot \mathbf{B})]\mathbf{R}\psi = 0. \quad (17)$$

In the next section, we apply Eq. (17) to AB problem, giving a focus to spin effects through the $e(\mathbf{S} \cdot \mathbf{B})$ term. We shall see later that by carefully modulating the radial operator (17) with boundary conditions, it can provide both bound and scattering states. However, we emphasize only bound states.

4 The Aharonov-Bohm problem

Let us consider the particular case where the boson moves in the presence of the AB potential ($A_0 = 0$). The vector potential in the Coulomb gauge is

$$e\mathbf{A} = -\frac{\phi}{\rho}\hat{\phi}, \quad (18)$$

where ϕ is the flux parameter. The potential in (18) provides a magnetic field perpendicular to the plane (ρ, ϕ) , namely

$$e\mathbf{B} = -\phi\frac{\delta(\rho)}{\rho}\hat{\mathbf{z}}, \quad (19)$$

where \mathbf{B} is the magnetic field due to a solenoid. If the solenoid is extremely long, the field inside is uniform, and the field outside is zero. However, the boson is allowed to access the $\rho = 0$ region. In this region, the magnetic field is non-null. If the radius of the solenoid is $\rho_0 \approx 0$, then the relevant magnetic field is $B \sim \delta(\rho)$ as in (19).

4.1 Aharonov-Bohm problem for the spin-1 sector

Now, we consider the effect of Aharonov-Bohm flux field on vector bosons. Substituting Eq. (18) in Eq. (17), we obtain

$$\left[\left(\frac{1}{i} \nabla + \frac{\phi}{\rho} \hat{\phi} \right)^2 + \phi S \frac{\delta(\rho)}{\rho} \right] \mathbf{R}\psi = (E^2 - M^2) \mathbf{R}\psi, \quad (20)$$

where S is the matrix

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (21)$$

and

$$\nabla = \hat{\rho} \frac{\partial}{\partial \rho} + \hat{\phi} \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{z} \frac{\partial}{\partial z}, \quad (22)$$

is the gradient operator in cylindrical coordinates. At this stage, we can use the invariance under boosts along the z -direction and adopt the usual decomposition

$$\mathbf{R}\psi = R^i \psi(\rho, \phi, z) = f_m^{(i)}(\rho) e^{im\phi} e^{ip_z z}, \quad (23)$$

with $m \in \mathbb{Z}$. Inserting Eq. (23) into Eq. (20), we get

$$\mathfrak{D} f_m^{(i)}(\rho) = k^2 f_m^{(i)}(\rho), \quad (24)$$

with $k = \sqrt{E^2 - M^2 - p_z^2}$, where

$$\mathfrak{D} = \mathfrak{D}_0 + \phi s^i \frac{\delta(\rho)}{\rho}, \quad (25)$$

$$\mathfrak{D}_0 = -\frac{d^2}{d\rho^2} - \frac{1}{\rho} \frac{d}{d\rho} + \frac{(m + \phi)^2}{\rho^2}, \quad (26)$$

and $s^i = (1, -1, 0)$ represents the eigenvalues of the operator S acting on the DKP spinor $f_m^{(i)}(\rho)$. Equation (24) describes the quantum dynamics of vector bosons in the presence of the Aharonov-Bohm potential. From (24) we can see that scattering states occur only if $k \in \mathbb{R}$, whereas bound states occur only if $k = i|k|$.

At this level, it is worthwhile to note that the solution for this problem can be separated in two cases. The first case is when the spin projection $s^3 = 0$. In this case the Pauli term is absent and the radial operator \mathfrak{D} becomes \mathfrak{D}_0 and $f_m^{(3)}$ can be expressed as a solution of the modified Bessel differential equation. We can see that the system does not admit bound-state solutions. On the other hand, the second case is when $s^{1,2} = 1, -1$. In this case the radial operator \mathfrak{D} is equivalent to Eq. (29) of Ref. [62] (see also Refs. [14, 63, 64]) which governs the quantum dynamics of the usual spin-1/2 AB problem, namely

$$-\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{(m + \phi)^2}{r^2} + \phi s \frac{\delta(r)}{r}, \quad (27)$$

where s is twice the spin value, with $s = +1$ for spin “up” and $s = -1$ for spin “down”, and ϕ is the flux parameter.

In order to study the dynamics of the system for $s^{1,2} = 1, -1$, it is necessary to solve Eq. (24). However, as for the case studied in Ref. [62], it also involves a singularity in the $r = 0$ region. The appropriate method for studying this problem is the self-adjoint extension method of operators in quantum mechanics. By exploiting the nature of the δ -function in Eq. (25), we can see that the system admits at least a bound state. In addition, we also must verify that the operator \mathfrak{D}_0 in Eq. (26) admits or not self-adjoint extensions. It is known that when we consider Eq. (25), that is, taking into account the δ -function, \mathfrak{D}_0 is not essentially self-adjoint. In this case, we must find the self-adjoint extensions of the operator \mathfrak{D}_0 corresponding to different types of boundary conditions. Such self-adjoint extensions are based in boundary conditions at the origin and conditions at infinity [65–67]. From the theory of symmetric operators, it is a well-known fact that the symmetric radial operator \mathfrak{D}_0 is essentially self-adjoint if

$$|m + \phi| \geq 1, \quad (28)$$

while for

$$|m + \phi| < 1, \quad (29)$$

it admits an one-parameter family of self-adjoint extensions [33], $\mathfrak{D}_{0, \zeta_m^i}$, where ζ_m^i is the self-adjoint extension parameter. According to Ref. [62], the operator (26) admits an one-parameter family of self-adjoint extensions. To characterize this parameter family of self-adjoint extension, we use the approaches proposed by Kay-Studer (KS) [68] and Bulla-Gesztesy (BG) [69], being both based on boundary conditions. In short, in the KS approach, the boundary condition is a match of the logarithmic derivatives of the zero-energy solutions for Eq. (24) and the solutions for the problem \mathfrak{D}_0 plus self-adjoint extension. In the BG approach, however, the boundary condition is a mathematical limit allowing divergent solutions for the operator \mathfrak{D}_0 in Eq. (26) at isolated points, provided they remain square integrable. Then, following Ref. [62], the energy spectrum using the KS approach is found to be

$$(E_m^i)^2 = M^2 - \frac{4}{\rho_0^2} \left[\left(\frac{\phi s^i + |m + \phi|}{\phi s^i - |m + \phi|} \right) \frac{\Gamma(1 + |m + \phi|)}{\Gamma(1 - |m + \phi|)} \right]^{\frac{1}{|m + \phi|}} + p_z^2, \quad (30)$$

where ρ_0 is a finite very small radius. This radius may be understood as a kind of physical regularization for the δ -function in Eq. (25). Moreover, according to the BG method, the energy spectrum is given by

$$(E_m^i)^2 - M^2 = -4 \left(-\frac{1}{\zeta_m^i} \frac{\Gamma(1 + |m + \phi|)}{\Gamma(1 - |m + \phi|)} \right)^{\frac{1}{|m + \phi|}} + p_z^2. \quad (31)$$

We can note in Eq. (30) that the KS method gives us energy levels without any arbitrary parameter that can come from it, while the BG method gives an expression that leaves an arbitrary parameter, namely, the self-adjoint extension of the parameter ζ_m^i . However, if we directly compare Eqs. (30) and (31), an expression for the self-adjoint extension parameter ζ_m^i is exactly found, i.e.,

$$\zeta_m = -\rho_0^{2|m+\phi|} \left(\frac{\phi s^i - |m+\phi|}{\phi s^i + |m+\phi|} \right). \quad (32)$$

The component of the DKP spinor for bound-state solutions for the spin projection $s^{1,2} = 1, -1$ is given by

$$R^i \psi(\rho, \phi, z) = C_m K_{|m+\phi|} \left(|k^{(i)}| \rho \right) e^{im\phi} e^{ip_z z}, \quad (33)$$

where C_m is a normalization constant, $K_{|m+\phi|}$ are the modified Bessel functions of second kind and $|k^{(i)}| = \sqrt{M^2 + p_z^2 - (E_m^i)^2}$ can be obtained from Eq. (30). This important result was found for the first time in Ref. [64], where was proposed a general regularization procedure to obtain the self-adjoint extension parameter for both state bound and scattering problem for the spin-1/2 AB problem in conical space in (1+2) dimensions. To ensure that Eq. (31) is a real number, the self-adjoint extension parameter must be negative, i.e., $\zeta_m^i < 0$. This condition, however, ensures that the system admits relativistic bound states.

5 Conclusions

In this work, we have addressed the spin-1 AB problem in the context of the DKP formalism. We have assumed vector bosons incidents on a flux tube, characterized by a δ -function. We found that our problem can be separated in two cases depending on the spin projection. For $s^3 = 0$, the Pauli's term is absent (δ -function is absent), the radial operator can be expressed as a modified Bessel differential equation and we can see that the system does not admit bound-state solutions. Otherwise, for $s^{1,2} = 1, -1$, the radial operator is equivalent to usual spin-1/2 AB problem and consequently, equivalent to the problem of a particle in the presence of a δ -function potential in one dimension in quantum mechanics. The easiest way of dealing with singularities in quantum mechanics is by imposing that the eigenfunction vanishes at the singularity. However, although convenient, this does not necessarily give the best description (or even the correct one) of the physical phenomenon studied. Care must be taken to insure that the operator is self-adjoint in the region of interest. This can be achieved by extending the domain of the operator \mathfrak{D}_0 in Eq. (26) to equal that of \mathfrak{D}_0^\dagger . By doing this, a family of boundary conditions might appear, including the one that requires the eigenfunction to vanish at the singularity. With this in hands, physics itself determines

the appropriate boundary condition. In this sense, the self-adjoint extension approach was used to determine the bound states of vector bosons for the spin projection $s^{1,2} = 1, -1$ in terms of the physics of the problem, in a very consistent way and without any arbitrary parameter. Finally, expressions for the bound states energy for vector bosons in the presence of the AB potential has been obtained and the conditions in which they occur were established.

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